

# SOME INTEGER FORMULA-ENCODINGS AND RELATED ALGORITHMS

EDINAH K. GNANG \* AND PATRICK DEVLIN

**ABSTRACT.** We investigate the special class of formulas made up of arbitrary but finite combinations of addition, multiplication, and exponentiation gates. The inputs to these formulas are restricted to the integral unit 1. In connection with such formulas, we describe two essentially distinct families of canonical formula-encodings for integers, respectively deduced from the decimal encoding and the fundamental theorem of arithmetic. Our main contribution is the detailed description of two algorithms which efficiently determine the canonical formula-encodings associated with relatively large sets of consecutive integers.

## 1. INTRODUCTION

It is a well known fact that the binary encoding is on average optimal for representing integers. However if we think of a binary string as a computer program, it follows that such a program implicitly describes a circuit which evaluates to the corresponding integer. When literally interpreted, the binary representation describes a sum of powers of two with the powers determined by the location of the bits. The recursive encoding which explicitly describes the circuit representation associated with the literal interpretation of the decimal strings was pioneered by Goodstein [4]. In the current discussion we depart slightly from conventional arithmetic circuit models [1, 2] in the fact that we consider circuits or more specifically formulas which combine fan-in two exponentiation, multiplication, and addition gates with input restricted to the integral unit 1. While at first it might seem unnatural to allow exponentiation gates, we argue that exponentiation gates are implicit in the decimal encoding. Furthermore, exponentiation gates are critical for obtaining a small circuit which evaluate to the integer specified by the input binary strings. Throughout the discussion the arithmetic formulas will be described with symbolic expressions and for convenience we associate with the symbol  $x$  the recurring formula  $(1 + 1)$ . Our main contribution is an asymptotically optimal algorithm for finding Goodstein formula encodings for relatively large subset of consecutive integers. Finally we describe an alternative canonical formula-encoding conjecturely smaller on average when compared with the Goodstein formula-encoding. We also provide an efficient algorithm for computing the latter formula-encodings for relatively large subsets of consecutive integers.

## 2. THE SET OF FORMULA-ENCODINGS OF POSITIVE INTEGERS

Let  $\mathcal{E}$  denote the set of symbolic expressions which result from finite combinations of addition, multiplication and exponentiations where the *only* input is 1. For instance (abbreviating, for convenience,  $x = 1 + 1$ )

$$(1) \quad x^{\wedge}(1 + (x \cdot x \cdot x) + x^{\wedge}(x \cdot x)) + 1^{\wedge}(1^{\wedge}1) + 1^{\wedge}(x^{\wedge}1) + 1^{\wedge}(x^{\wedge}(x^{\wedge}1)) + 1 \in \mathcal{E}.$$

Elements of the set  $\mathcal{E}$  for our purposes will be encoded as strings from the alphabet  $\mathfrak{A}$

$$(2) \quad \mathfrak{A} := \{1, +, \cdot, ^{\wedge}\}.$$

For the reader's convenience we shall adopt the infix notation thereby making use of the parenthesis characters '(' and ')'. However we point out that the parenthesis characters '(' and ')' can be omitted from the alphabet  $\mathfrak{A}$  since either the postfix or prefix notations avoid their use entirely.

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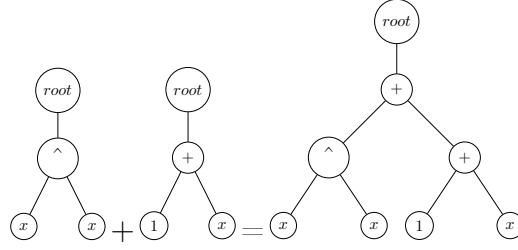


FIGURE 1. Illustration of a formula operation.

Of course every such expression evaluates to a positive integer, and we are interested in the shortest possible expression of representing any given positive integer, or at least (for large integers) as close as possible. The evaluation function is defined recursively as

$$(3) \quad E(a + b) = E(a) + E(b) \quad , \quad E(a \cdot b) = E(a) \cdot E(b) \quad , \quad E(a^b) = E(a)^{E(b)}.$$

One can introduce “axioms” that transform one tree to another without changing its value, but these are left to the reader.

### 3. CANONICAL FORMS

We shall crucially require for our purposes the notion of *canonical form expressions* or *canonical form formulas*. Canonical form expressions or formulas are elements of  $\mathcal{E}$  which we think of as unambiguous representatives of the corresponding integer. We will discuss here two important canonical forms. We point out however that our choices of canonical forms are bound to be somewhat arbitrary and incidentally alternative representative choices could be made.

**3.1. The First Canonical Form.** An expression  $f \in \mathcal{E}$  is in the First Canonical Form (FCF) if  $f$  corresponds to a finite sum of the form (recall that  $x$  is short for  $1 + 1$ )

$$(4) \quad f = \sum_k (x^{\wedge} f_k) \text{ or } f = 1 + \sum_k (x^{\wedge} f_k)$$

such that the expressions  $f_k$  are distinct for distinct values of the index  $k$  and each one of the expressions  $f_k \in \mathcal{E}$  being themselves in the FCF.

**Proposition 1:** An arbitrary  $f \in \mathcal{E}$  is either in the FCF or can be transformed into an expression in the FCF via a finite sequence of transformations which preserve evaluation value.

**Proposition 2:** Every expression  $f \in \mathcal{E}$ , is a member of finite set of non trivial equivalent expressions (i.e. expressions not including any subexpressions of the form  $g^{\wedge} 1$ ,  $1^{\wedge} g$  or  $g \cdot 1$  for some arbitrary expression  $g$ ).

**Proof :** A constructive proof of proposition 1 and 2 readily follows from the quotient remainder theorem.

**3.2. The Second Canonical Form.** An expression  $f \in \mathcal{E}$  is in the Second Canonical Form (SCF) if  $f$  corresponds to a finite product of the form

$$(5) \quad f = \prod_k [(1 + f_k)^{\wedge} g_k] \text{ or } f = (x^{\wedge} g) \cdot \prod_k [(1 + f_k)^{\wedge} g_k]$$

where for distinct values of the index  $k$ , the formula associated with  $(1 + f_k)$  encodes distinct primes greater than 2. Furthermore,  $f_k, g_k, g \in \mathcal{E}$  are themselves expressions in the SCF.

**Proposition 3:** An arbitrary  $f \in \mathcal{E}$  is either in the SCF or can be transformed into an expression in the SCF via a finite sequence of transformations which preserve the evaluation value.

**Proof :** A constructive proof of proposition 3 immediately follows from the fundamental theorem of arithmetic.

A considerable advantage of the SCF as a default encoding for integers is the fact that the encoding considerably simplifies the computational complexity analysis of formulas arithmetic. In particular the complexity analysis of formulas arithmetic (multiplication and exponentiation) reduces to the analysis of the formula addition operations. Furthermore it has been empirically observed that the lengths of expressions describing formulas in the SCF have smaller expected length than their FCF counterpart. We further remark that the SCF is implicit in the discussion of integer prime tower encodings [3].

#### 4. COMPUTING FCF INTEGER ENCODINGS

We describe here an asymptotically optimal algorithm for determining symbolic expressions which describe FCF integer formula-encoding for relatively large set of consecutive integers. The algorithm is based on the observation that given the FCF encoding of the first  $n$  positive integers one easily deduces from them the FCF encoding for the next  $2^n - n$  positive integers. We pointed out earlier that the FCF encoding describes formulas corresponding to the Goodstein base 2 recursive (or hereditary) integer encoding [4], however it is clear that an attempt to uncover the FCF encoding of integers by simply iterating through consecutive integers and recursively expressing in binary form the powers of 2, would yield a very inefficient algorithm. Incidentally our proposed algorithm for determining FCF integer formula-encodings amounts to a set recurrence. The initial sets for the recurrence is specified by

$$\mathbb{N}_0 := \{1\}$$

and the set recursion is defined by

$$\mathbb{N}_{k+1} = \bigcup_{S \in \{\{1\} \cup x^{\mathbb{N}_k}\}} \left\{ \sum_{s \in S} s \right\}.$$

where for an arbitrary symbolic expression  $f$  and a set of symbolic expressions  $L$ , the set  $f^L$  is to be interpreted as:

$$\{f^l\}_{l \in L}.$$

Three iterations of the set recurrence yield

$$\text{FCF}_3 := \left\{ 1, x, (x+1), x^x, (x^x+1), \dots, (x^x + x^{(x^x)} + x + x^{(x^x+x+1)} + x^{(x^x+x)} + x^{(x^x+1)} + x^{(x+1)} + 1) \right\}.$$

It follows from the definition of the set recurrence, that the proposed algorithm requires  $O(\log_\star(n))$  iterations to produce FCF formulas for all positive integers less than  $n$  and the algorithm requires optimally  $O(n)$  symbolic expression manipulations.

#### 5. ZETA RECURSION.

We recall here an elementary recursion called the *Zeta recursion* emphasizing its close resemblance with the Zeta summation formula. Let us briefly recall here the *Zeta recursion* first introduced in [3] as a combinatorial construction for sifting primes.

$$(6) \quad \mathbb{P}_0 := \{x\}, \check{\mathbb{N}}_0 := \mathbb{P}_0 \cup \{1\}$$

we consider the set recurrence relation defined by

$$(7) \quad \mathbb{N}_{k+1} = \prod_{p \in \mathbb{P}_i} \left\{ \{1\} \cup p^{\check{\mathbb{N}}_k} \right\},$$

where

$$p^{\check{\mathbb{N}}_k} := \{p^n \text{ such that } n \in \check{\mathbb{N}}_k\},$$

and for sets of symbolic expressions  $\{S_i\}_{0 \leq i < m}$

$$\prod_{0 \leq i < m} S_i := \left\{ \prod_{0 \leq k < m} s_k \right\}_{s_i \in S_i}.$$

Finally  $\check{\mathbb{N}}_{k+1}$  is deduced from  $\mathbb{N}_{k+1}$  by adjunction of missing primes suggested by identification of gaps of size two between consecutive elements of  $\mathbb{N}_{k+1}$ , hence

$$(8) \quad \mathbb{P}_{k+1} = \mathbb{P}_k \cup (\check{\mathbb{N}}_{k+1} \setminus \mathbb{N}_{k+1}),$$

so that for all  $k \geq 0$ , we have  $\mathbb{P}_k \subsetneq \check{\mathbb{N}}_k$  and

$$(9) \quad \begin{cases} \check{\mathbb{N}}_k \subsetneq \check{\mathbb{N}}_{k+1} \\ \mathbb{P}_k \subsetneq \mathbb{P}_{k+1} \end{cases}.$$

Furthermore we can use the *Zeta recursion* to iteratively construct larger and larger subsets of rational numbers deducing the set  $\mathbb{Q}_k$  from the previously obtained sets  $\check{\mathbb{N}}_k$  and  $\mathbb{P}_k$  as follows

$$(10) \quad \mathbb{Q}_k = \prod_{p \in \mathbb{P}_k} \left\{ \left( \frac{1}{p} \right)^{\check{\mathbb{N}}_k} \cup \{1\} \cup p^{\check{\mathbb{N}}_k} \right\},$$

where

$$(11) \quad \mathbb{Q}_k \subsetneq \mathbb{Q}_{k+1}.$$

Which yields an alternative combinatorial proof of Cantor's result establishing that the rational numbers are countable.

**5.1. Improved Zeta recursion.** Some slight modifications to the *Zeta recursion* has the benefit of improving the computational performance of the recurrence computation.

$$(12) \quad \mathbb{P}_0 := \{x\}, \check{\mathbb{N}}_0 := \mathbb{P}_0 \cup \{1\}$$

for an arbitrary  $q \in \mathbb{P}_k$  we have

$$(13) \quad \mathbb{N}_{q,k+1} = \bigcup_{\substack{n \in \check{\mathbb{N}}_k \\ q^n < 2^{k+2}}} \left\{ [2^{k+1}, 2^{k+2}] \cap \left( q^n \times \prod_{\substack{p \in \mathbb{P}_k \\ p < q}} \{ \{1\} \cup p^{\check{\mathbb{N}}_k} \} \right) \right\}$$

from which we have that

$$(14) \quad \mathbb{N}_{k+1} = \check{\mathbb{N}}_k \cup \left( \bigcup_{q \in \mathbb{P}_k} \mathbb{N}_{q,k+1} \right).$$

The completion of the set  $\mathbb{N}_{k+1}$  to  $\check{\mathbb{N}}_{k+1}$  is still determined by sorting the element in the set  $\bigcup_{q \in \mathbb{P}_k} \mathbb{N}_{q,k+1}$  and adjoining missing primes located by identifying gaps of size two between consecutive elements of  $\mathbb{N}_{k+1}$ .

The improved Zeta recursion is not a particularly efficient algorithm for the sole purpose of siefting primes because it implicitly requires us to store rather large list of integers. The algorithm is however particularly well suited to the task of determining symbolic expressions describing SCF formula-encoding for a relatively large set of consecutive integers, with initial sets for the iteration being

$$\mathbb{P}_0 := \{x\}, \check{\mathbb{N}}_0 := \mathbb{P}_0 \cup \{1\}.$$

For instance seven iterations of the improved Zeta recursion yield

$$\text{SCF}_7 = \left\{ 1, x, (x+1), x^x, (x^x+1), (x+1)x, ((x+1)x+1), \dots, (x+1) \left( x^{(x^x)} + 1 \right) (x^x+1) \right\}$$

It follows from the definition of the Zeta recursion that the proposed algorithm requires  $O(\log n)$  iterations to determine SCF formulas for all positive integers less than  $n$  and the algorithm requires  $O\left(\frac{n^2}{\log n}\right)$  symbolic expression manipulations.

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EDINAH K. GNANG, COMPUTER SCIENCE DEPARTMENT, RUTGERS UNIVERSITY (NEW BRUNSWICK), PISCATAWAY, NJ, USA.

*E-mail address:* `gnang@cs.rutgers.edu`

PATRICK DEVLIN, MATHEMATICS DEPARTMENT, RUTGERS UNIVERSITY (NEW BRUNSWICK), PISCATAWAY, NJ, USA.

*E-mail address:* `prd41@math.rutgers.edu`